Overall Exam, Applied and Computational Mathematics part

Question I

Consider the ordinary differential equation (ODE) for $y(t) \in \mathbb{R}$ as

$$\varepsilon^2 y''(t) + \left(\mu^2 + \frac{1}{\varepsilon^2}\right) y(t) = f(t), \quad t > 0,\, y(0) = y_0,$$

where $\mu > 0$ is a constant, $f(t)$ is a given function, and $0 < \varepsilon \ll 1$ is a small parameter.

1. Use variation-of-constant formula to reformulate the ODE into an equivalent integral form.
2. Construct a second order numerical scheme for solving the ODE, whose accuracy is independent of ε.
3. Prove the convergence rates of the constructed numerical scheme above.

Question II

Show that the finite difference method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + u_j^n$$

for the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(-\infty < x < +\infty, t > 0)$ is stable if $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$.